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Thin-flame theory for a fuel droplet in slow viscous flow

By FRANCIS E. FENDELL, MAUREEN L. SPRANKLE AND DAVID S. DODSON

TRW Systems, Redondo Beach, California

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The equilibrium burning of a spherical drop of pure non-gaseous fuel in a slow convective flow of hot oxidant is examined for Lewis number unity. A Stokes or Oseen flow with modified boundary conditions to permit mass transfer at the drop surface describes the velocity field. The method of inner and outer expansions is then adopted to describe the thermal and mass-fraction profiles under the model of a direct one-step irreversible indefinitely fast chemical reaction. The thin-flame position and surface mass-transfer rate, both functions of polar angle as well as radial position when convection is added to the conventional diffusive transport, are furnished in terms of the Peclet number. It is found that the interaction of the perturbational free-streaming with the asymmetric vaporization it induces can lead to drag coefficients in excess of the Stokes value.

1. Introduction

The vaporization of a spherical fuel droplet in an infinite expanse of oxidant and the subsequent homogeneous combustion is examined for low-Reynolds-number flow. An irreversible, indefinitely fast, one-step chemical reaction is adopted (Burke & Schumann 1928) so the burning zone collapses to a mathematical interface where the reactants meet in stoichiometric proportion. Of particular interest is how small convective transport augments the purely diffusively controlled mass-transfer rate of previous analyses (Godsave 1953; Goldsmith & Penner 1954).

Williams (1960) has indicated that quasi-steady vaporization at unit Lewis number, with the droplet temperature just below boiling, serves as an excellent approximation. Furthermore, surface-tension effects, droplet internal circulation, and droplet shape distortion are here negligible. For analytic convenience incompressibility and constant fluid properties are adopted.

The conventional non-dimensionalized formulation of the problem after Schvab & Zeldovich (Williams 1965) is

$$\rho = \text{const.}, \nabla . \mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial x} = -\frac{\nabla \times (\nabla \times \mathbf{u})}{R} - \nabla P, \tag{2}$$

$$L(\overline{Y}_O + \overline{T}) = L(\overline{Y}_F - \overline{Y}_O) = 0, \tag{3}$$

where $x = r\cos\theta$ (see figure 1) and, if $\epsilon = U_\infty/aD$,

$$L = \epsilon\left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right) - \nabla^2 \quad (\epsilon \ll 1) \tag{4}$$

by azimuthal symmetry. The boundary conditions are at $r \to \infty$:

$$\bar{Y}_O \to (\bar{Y}_O)_\infty, \quad \bar{T} \to \bar{T}_\infty, \quad \mathbf{u} \to \hat{\imath} \tag{5}$$

and at $r = 1$:

$$\frac{\partial \bar{T}(1,\theta,\epsilon)}{\partial r} = [\epsilon u_r(1,\theta,\epsilon)]\,\bar{L}m, \tag{6}$$

$$\frac{\partial \bar{Y}_F(1,\theta,\epsilon)}{\partial r} = -[\epsilon u_r(1,\theta,\epsilon)]\,[\alpha_F - \bar{Y}_F(1,\theta,\epsilon)] \tag{7}$$

$$u_\theta(1,\theta,\epsilon) = 0, \tag{8}$$

$$\bar{Y}_F(1,\theta,\epsilon) = \alpha_F \exp\left\{\bar{\chi}\left[\frac{1}{\bar{T}_B} - \frac{1}{\bar{T}(1,\theta,\epsilon)}\right]\right\}. \tag{9}$$

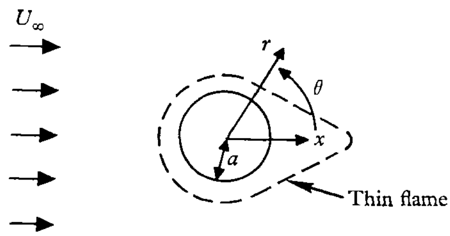

FIGURE 1. Co-ordinate system for flow past a spherical droplet.

Equation (6) is an adiabatic vaporization condition; (7), a statement that the droplet remains pure fuel; and (9), the Clausius–Clapeyron equation. Initially (9) is replaced by

$$\bar{T}(1,\theta,\epsilon) = \bar{T}_a(\theta,\epsilon). \tag{9a}$$

The radial velocity at the droplet surface $u_r(1,\theta,\epsilon)$ is to be specified in the course of solution and is in this sense an eigenvalue. According to the Burke–Schumann model

$$\bar{Y}_F = 0 \quad (r_* \leqslant r \leqslant \infty), \tag{10}$$

$$\bar{Y}_O = 0 \quad (1 \leqslant r \leqslant r_*), \tag{11}$$

where $r_*(\theta,\epsilon)$ is to be determined by applying (10) and (11) to the solution of (3).

The technique invoked here is that of inner and outer expansions, used by Proudman & Pearson (1957) to obtain the velocity field and by Acrivos & Taylor (1962) to obtain the temperature field for slow viscous flow past an isothermal sphere. However, finite interfacial velocity has not previously been carefully examined in relation to heat and mass transfer at low Reynolds number.

The procedures developed here extend directly to many other cases of heat and mass transfer for small droplets in slow flow, such as the growth of raindrops by condensation or their decay by evaporation in an almost steady atmosphere.

A partial list of symbols used throughout this article is given below:

- b — stoichiometric coefficient of oxidant;
- D — mass-transfer coefficient in Fick's first law;
- d — stoichiometric coefficient of fuel;
- E_n — a sequence of eigenvalues related to mass transfer [see equation (16)];
- E_{mn} — another sequence of eigenvalues related to E_n [see equation (25)];
- ΔH_c — specific heat released through combustion: $m_O b h_0^O + m_F d h_0^F - m_P h_0^P p$;

h_0^O, h_0^F, h_0^P specific heats of formation of oxidant, fuel, and product, respectively;
$\bar{L}$ $L/\Delta H_c$, where L is the specific heat of vaporization of fuel;
m $bm_O + dm_P$;
m_F, m_O, m_P molecular weight of fuel, oxidant, and product, respectively;
P pressure;
p stoichiometric coefficient of product;
$P_m(\mu)$ Legendre polynomial of degree m;
$\bar{p}$ $[\bar{T} - \bar{T}_\infty + \bar{Y}_O - (\bar{Y}_O)_\infty]/[\bar{T}_{00} - \bar{T}_\infty - (\bar{Y}_O)_\infty]$;
$\overline{p_n}$ terms in the series for $\overline{p^i}$;
$\overline{P_n}$ terms in the series for $\overline{p^o}$;
$\bar{q}$ $\bar{Y}_F - \bar{Y}_O + (\bar{Y}_O)_\infty$;
R Reynolds number; here $U_\infty a/\nu$;
$\bar{T}$ $T/(\Delta H_c/mc_P)$, non-dimensionalized temperature;
u. $(u_r, u_\theta, 0)$ macroscopic net velocity (non-dimensionalized by U_∞);
$\bar{Y}_F$ $[m/(dm_F)]Y_F$, stoichiometrically adjusted mass fraction of fuel;
$\bar{Y}_O$ $[m/(bm_O)]Y_O$, stoichiometrically adjusted mass fraction of oxidant.

Greek symbols

ϵ Peclet number; here $U_\infty a/D$;
μ $\cos\theta$;
ρ density; also strained radial co-ordinate ($\rho = \epsilon r$);
$\bar{\chi}$ $(\bar{L}mm_F c_P/\mathrm{R})$ where here only R is the universal gas const.

Superscripts and subscripts

— non-dimensionalized or stoichiometrically adjusted;
F fuel;
i, o inner or outer expansion;
O oxidant;
$*$ evaluated at the thin flame.

2. Flow

If $\mathbf{u} = \nabla \times \left[\dfrac{\psi(r,\theta)}{r\sin\theta}\right]\hat{\phi}$, then (2) becomes

$$R\left[\frac{1-\mu^2}{r}\frac{\partial(D_r^2\psi)}{\partial\mu} + \mu\frac{\partial(D_r^2\psi)}{\partial r}\right] = D_r^4\psi, \tag{12}$$

where $\mu = \cos\theta$ and

$$D_r^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\psi}{\partial\mu^2}. \tag{13}$$

If $R \to 0$,

$$\psi(r,\mu) = \sum_{n=0}^{\infty}[(A_n r^{n+3} + B_n r^{n+1} + C_n r^{-n} + D_n r^{-n+2}).Q_n(\mu)], \tag{14}$$

where $Q_n(\mu) = \int_{-1}^{\mu} P_n(\mu_1)\,d\mu_1$, in which $P_n(\mu)$ signifies the Legendre polynomial of degree n. If

$$\mu_r(1,\theta) = -\frac{\partial\psi(1,\mu)}{\partial\mu} = \sum_{m=0}^{\infty} E_m P_m(\mu) \tag{15}$$

then application of equations (5), (8), and (15) to (14) gives

$$-\psi(r,\mu) = +E_0 Q_0 + \left(r^2 + \frac{E_1+1}{2r} + \frac{E_1-3}{2} r\right) Q_1 + \sum_{n=2}^{\infty} \tfrac{1}{2} n (E_n) \left(\frac{2-n}{n} r^{-n} + r^{2-n}\right) Q_n. \tag{16}$$

3. Inner and outer expansions

Use of equation (16) in (3) yields [$\overline{p}$ signifies either $(\overline{Y}_O + \overline{T})$ or $(\overline{Y}_F - \overline{Y}_O)$]

$$\begin{aligned}\epsilon \Bigg(\Bigg[\frac{E_0}{r^2} P_0(\mu) &+ \left(1 + \frac{E_1-3}{2r} + \frac{E_1+1}{2r^3}\right) P_1(\mu) \\ &+ \sum_{n=2}^{\infty} \tfrac{1}{2} n E_n \left(r^{-n} + \frac{2-n}{n} r^{-2-n}\right) P_n(\mu)\Bigg] \frac{\partial \overline{p}}{\partial r} \\ &+ \Bigg\{ \left(-\frac{E_1-3}{2r^2} + \frac{E_1+1}{2r^4} - \frac{2}{r}\right) \left[\frac{P_2(\mu) - P_0(\mu)}{3}\right] \\ &+ \sum_{n=3}^{\infty} \frac{(2-n)n}{2(2n+1)} E_n [P_{n+1}(\mu) - P_{n-1}(\mu)] (r^{-n-3} - r^{-n-1}) \Bigg\} . \frac{\partial \overline{p}}{\partial \mu} \Bigg) \\ = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \overline{p}}{\partial r}\right) &+ \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[(1-\mu^2) \frac{\partial \overline{p}}{\partial \mu}\right].\end{aligned} \tag{17}$$

The inner expansion for $\overline{p}(r,\mu,\epsilon)$ emphasizes the dominance of molecular transport near the sphere

$$\overline{p^i}(r,\mu,\epsilon) = \sum_{n=0}^{\infty} f_n(\epsilon) \overline{p}_n(r,\mu), \tag{18}$$

while the outer expansion expresses the equal roles of convection and conduction far from the sphere

$$\overline{p^o}(r,\mu,\epsilon) = \sum_{n=0}^{\infty} g_n(\epsilon) \overline{P}_n(\rho,\mu) \quad (\rho = \epsilon r). \tag{19}$$

Equation (9*a*) is written ($\overline{T}_{mn}$ are regarded as given)

$$\overline{T}(1,\mu,\epsilon) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \overline{T}_{mn} P_m(\mu) f_n(\epsilon). \tag{20}$$

For convenience $\overline{p}$ may be re-defined as

$$\overline{p} = \frac{\overline{T} - \overline{T}_\infty + \overline{Y}_O - (\overline{Y}_O)_\infty}{\overline{T}_{00} - \overline{T}_\infty - (\overline{Y}_O)_\infty}. \tag{21}$$

At $r \to \infty$ the boundary conditions require that $\overline{P}_n = 0$ for all n. Also, $\overline{Y}_O = 0$ at $r = 1$ for parameter values of interest. Since once can anticipate

$$\overline{p}_n(r,\mu) = \sum_{m=0}^{\infty} h_{mn}(r) P_m(\mu) \tag{22}$$

the boundary condition (20) becomes

$$h_{mn}(1) = \frac{\overline{T}_{mn}}{\overline{T}_{00} - \overline{T}_\infty - (\overline{Y}_O)_\infty} \quad (n \geqslant 1). \tag{23}$$

One can also anticipate $f_0(\epsilon) = 1$ and $h_{m0} = \overline{T}_{m0} = 0$ for $m > 0$; i.e. the spherically symmetric result of Godsave is recovered for the lowest-order inner solution. Hence

$$h_{00}(1) = 1, \quad h_{m0}(1) = 0 \quad (m \neq 0). \tag{24}$$

Further, it is adopted that

$$E_m(\epsilon) = \sum_{n=0}^{\infty} E_{mn}[f_n(\epsilon)/\epsilon] \tag{25}$$

so from (15)

$$\epsilon u_r(1,\mu,\epsilon) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} E_{mn} P_m(\mu) f_n(\epsilon). \tag{26}$$

It should be noted that only the group $\epsilon u_r(1,\mu,\epsilon)$ appears in the boundary-value problem. In conjunction with previous anticipations, $E_{m0} = 0$ for $m > 0$ so

$$\epsilon u_r(1,\mu,\epsilon) = E_{00} + o(1). \tag{27}$$

The eigenvalue $u_r(1,\mu,\epsilon)$ is not singular as $\epsilon \to 0$; the dimensional form of (27) is

$$\frac{v_r(1,\mu,\epsilon)}{(D/a)} = E_{00} + o(1). \tag{28}$$

Equation (6) becomes

$$\frac{dh_{mn}(1)}{dr} = \frac{m\overline{L}}{\overline{T}_{00} - \overline{T}_\infty - (\overline{Y}_O)_\infty} E_{mn}. \tag{29}$$

4. Lowest-order outer solution

If $\rho = \epsilon r$ is substituted into (17), $\overline{P}_0$ is governed by an equation involving convection by the undisturbed freestream

$$\nabla_\rho^2 \overline{P}_0 = \mu \frac{\partial \overline{P}_0}{\partial \rho} + \frac{1-\mu^2}{\rho}\frac{\partial \overline{P}_0}{\partial \mu}. \tag{30}$$

Following Lamb (1945),

$$\overline{P}_0(\rho,\mu) = G_0(\rho,\mu)\, e^{\frac{1}{2}\rho\mu}, \tag{31}$$

or

$$(\nabla_\rho^2 - \tfrac{1}{4})\, G_0 = 0. \tag{32}$$

By separation of variables and the requirement of boundedness at infinity

$$G_0(\rho,\mu) = \left(\frac{\pi}{\rho}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} C_k K_{k+\frac{1}{2}}(\tfrac{1}{2}\rho) P_k(\mu), \tag{33}$$

where $K_{k+\frac{1}{2}}(\frac{1}{2}\rho)$ is a half-order Bessel function given by

$$K_{k+\frac{1}{2}}(\tfrac{1}{2}\rho) = \left(\frac{\pi}{\rho}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\rho} \sum_{m=0}^{k} \frac{(k+m)!}{(k-m)!\, m!\, \rho^m}. \tag{34}$$

As $\rho \to 0$, to lowest order

$$\overline{p^0} \to \frac{\pi g_0(\epsilon)}{\rho}[1 + \tfrac{1}{2}\rho(\mu - 1) + \ldots] \cdot \sum_{k=0}^{\infty} C_k \frac{(2k)!}{k!\,\rho^k} P_k(\mu). \tag{35}$$

The C_k and $g_0(\epsilon)$ are identified below by matching to the inner solution.

5. Lowest-order inner solution

For no mass transfer ($E_{mn} = 0$ for all m, n) Acrivos & Taylor (1962) found $f_0(\epsilon) = 1$ and $f_1(\epsilon) = \epsilon$. Under this ordering the lowest-order inner equation becomes, from equation (17),

$$\left(\nabla_r^2 - \frac{E_{00}}{r^2}\frac{\partial}{\partial r}\right)\bar{p}_0 = 0. \tag{36}$$

Seeking solution in accordance with (22),

$$r^2\frac{d^2h_{m0}}{dr^2} + (2r - E_{00})\frac{dh_{m0}}{dr} - m(m+1)h_{m0} = 0. \tag{37}$$

For $E_{00} \neq 0$, $r = 0$ is an irregular singular point. The method of Frobenius gives an indicial equation for solution in ascending powers of r about $r = 0$ to which there is only one root. The following polynomial arises:

$$h_{m0}^{(1)} = \sum_{n=0}^{m} a_n r^n, \tag{38}$$

$$a_0 = \frac{m!\,E_{00}^m}{\prod_{s=1}^{m}(m+s)\prod_{q=0}^{m-1}(q-m)}, \tag{39}$$

$$a_n = (E_{00})^{m-n}\left(\frac{m!}{n!}\right)\frac{\prod_{s=1}^{n}(m+s)\prod_{q=0}^{n-1}(q-m)}{\prod_{s=1}^{m}(m+s)\prod_{q=0}^{m-1}(q-m)} \quad (n \geqslant 1). \tag{40}$$

Explicitly,

$$h_{10} = r - \tfrac{1}{2}E_{00}, \quad h_{20} = r^2 - \tfrac{1}{2}E_{00}r + \tfrac{1}{12}E_{00}^2. \tag{41}$$

If

$$h_{m0} = V_m(r)\exp(-E_{00}/r),$$

$$r^2\frac{d^2V_m}{dr^2} + (2r + E_{00})\frac{dV_m}{dr} - m(m+1)V_m = 0 \tag{42}$$

so the second set is

$$h_{m0}^{(2)} = \left(\sum_{n=0}^{m} b_n r^n\right)\exp(-E_{00}/r), \tag{43}$$

where

$$b_0 = (-)^m a_0, \quad b_n = (-)^{m-n}a_n \quad (n \geqslant 1). \tag{44}$$

When $E_{00} \to 0$ equation (36) reverts to Laplace's equation. Since the solution (38) goes to r^m, the second solution may be taken as

$$(-)^{m+1}\frac{2^{2m}(2m+1)}{E_{00}^{2m+1}}\prod_{n=1,3,\ldots}^{2m-1} n^2\left(\exp(-E_{00}/r)\sum_{n=0}^{m} b_n r^n - \sum_{n=0}^{m} a_n r^n\right) \tag{45}$$

if a form which goes to r^{-m-1} for $E_{00} = 0$ is desired. However, the special case $E_{00} = 0$ will be disregarded here and the form (43)–(44) adopted for the most part.

The solution to (36) subject to (24) results in

$$\bar{p}^i = A_0 + (1 - A_0)\frac{\exp(-E_{00}/r) - 1}{\exp(-E_{00}) - 1} + \sum_{m=1}^{\infty} A_m\left[h_{m0}^{(1)}(r) - \frac{h_{m0}^{(1)}(1)}{h_{m0}^{(2)}(1)}h_{m0}^{(2)}(r)\right]P_m(\mu) + O(\epsilon). \tag{46}$$

As $r \to \infty$ the term in brackets in (46) goes as r^m. To match (35) and (46) one takes $g_0(\epsilon) = \epsilon$, $A_m = 0$ for $m \geqslant 0$, $C_k = 0$ for $k > 0$ and

$$\pi C_0 = -E_{00}/\{\exp(-E_{00}) - 1\}. \tag{47}$$

The lowest-order inner solution is uniformly valid in r:

$$\overline{p^i} = \frac{\exp(-E_{00}/r) - 1}{\exp(-E_{00}) - 1} + O(\epsilon). \tag{48}$$

Clearly $E_{m0} = 0$ for $m \geqslant 1$ and

$$E_{00} = \ln\left[\frac{\overline{L}m + (\overline{Y}_O)_\infty + \overline{T}_\infty - \overline{T}_{00}}{\overline{L}m}\right]. \tag{49}$$

The classical spherically symmetric theory of Godsave (1953) has now been recovered.

The lowest-order outer solution is now known to be

$$\overline{p^o} = -\frac{\epsilon E_{00}}{\exp(-E_{00}) - 1}\,\frac{\exp\{\frac{1}{2}\rho(\mu - 1)\}}{\rho} + o(\epsilon), \tag{50}$$

which is the spherical source solution when uniform convection is present. The co-ordinate scaling has reduced the sphere to a point disturbance to lowest order.

When $E_{00} \to 0$ equation (48) shows that the lowest-order inner solution goes to $1/r$ and the lowest-order outer by (50) becomes $\rho^{-1}\exp[\frac{1}{2}\rho(\mu - 1)]$. These are the forms given by Acrivos & Taylor (1962) in the absence of mass transfer.

6. The next-order inner solution

Expanding the exponential in (48) and letting $\rho = \epsilon r$, one finds that none of the terms beyond the first can match to the solution (50), as expanded for small ρ

$$\overline{p^o} = -\frac{\epsilon E_{00}}{\exp(-E_{00}) - 1}\,\frac{1}{\rho}\,[1 + \tfrac{1}{2}\rho(\mu - 1) + \tfrac{1}{8}\rho^2(\mu - 1)^2 + O(\rho^3)] + O(\epsilon^2). \tag{51}$$

The next-to-lowest-order term of the inner expansion, which matches to the $(\mu - 1)$ term of (51), is now developed.

From equations (17), (22), and (48)—since $f_1 = \epsilon$ is anticipated—

$$\left.\begin{aligned} & r^2\frac{d^2h_{m1}}{dr^2} + (2r - E_{00})\frac{dh_{m1}}{dr} - m(m+1)h_{m1} \\ & = \begin{cases} [E_{01}E_{00}/(\exp(-E_{00}) - 1)]\,[\exp(-E_{00}/r)/r^2] & (m = 0), \\ [E_{00}/(\exp(-E_{00}) - 1)]\left[1 + \dfrac{E_{11} - 3}{2r} + \dfrac{E_{11} + 1}{2r^3}\right]\exp(-E_{00}/r) & (m = 1) \\ [E_{00}/(\exp(-E_{00}) - 1)]\,[\tfrac{1}{2}mE_{m1}]\left[r^{-m} + \dfrac{2 - m}{m}r^{-2-m}\right]\exp(-E_{00}/r) & (m \geqslant 2). \end{cases} \end{aligned}\right\} \tag{52}$$

It is simple to show on the basis of asymptotic forms for large r that, unless $h_{m1} = 0$ for $m \geqslant 2$, the asymptotic ordering of the inner series in ϵ is destroyed. The plausible result is that, just as the lowest-order inner solution involved only P_0, the first modification involves only P_0 and P_1.

By variation of parameters

$$h_{01} = A + B\frac{\exp(-E_{00}/r)-1}{E_{00}} + \frac{E_{01}}{\exp(-E_{00})-1} \times \left[\exp(-E_{00}/r)\left(1-\frac{1}{r}\right) + \frac{\exp(-E_{00})-\exp(-E_{00}/r)}{E_{00}}\right], \tag{53}$$

where A and B are constants of integration. Applying (23) to (53)

$$A + B\left(\frac{\exp(-E_{00})-1}{E_{00}}\right) = \frac{\overline{T}_{01}}{\overline{T}_{00}-\overline{T}_\infty-(\overline{Y}_O)_\infty}. \tag{54}$$

Matching (53) and (51)

$$A = \frac{1}{\exp(-E_{00})-1}\left[\frac{E_{00}}{2} - E_{01}\left(1+\frac{\exp(-E_{00})-1}{E_{00}}\right)\right]. \tag{55}$$

Applying (29) to (53)

$$B\exp(-E_{00})\,[\overline{T}_{00}-\overline{T}_\infty-(\overline{Y}_O)_\infty] = \overline{L}mE_{01}. \tag{56}$$

From (54) to (56)

$$E_{01} = \frac{E_{00}\exp(-E_{00})\{\overline{T}_{01}(\exp(-E_{00})-1)-\tfrac{1}{2}E_{00}[\overline{T}_{00}-\overline{T}_\infty-(\overline{Y}_O)_\infty]\}}{\overline{L}m(\exp(-E_{00})-1)^2-[1+(\exp(-E_{00})-1)/E_{00}]\,E_{00} \times\exp(-E_{00})\,[\overline{T}_{00}-\overline{T}_\infty-(\overline{Y}_O)_\infty]}. \tag{57}$$

Since $E_{00} > 0$ and $[\overline{T}_\infty + (\overline{Y}_O)_\infty] > \overline{T}_{00}$ usually, $E_{01} > 0$ necessarily if $\overline{T}_{01} < 0$. But, if $\overline{T}_{00} \to \overline{T}_B$, $\overline{T}_{01} < 0$. Thus the net effect of the first perturbation over all μ is to augment the mass transfer over the classical, purely diffusive value (see the Appendix).

If K_1 and K_2 are constants of integration, again by variation of parameters

$$h_{11} = [K_1+\alpha J_1(r)]\,(r-\tfrac{1}{2}E_{00}) + [K_2+\alpha J_2(r)]\exp(-E_{00}/r)\,(r+\tfrac{1}{2}E_{00}), \tag{58}$$

where
$$J_1(r) = \frac{4}{E_{00}^3}\left\{\exp(-E_{00}/r)\left(\tfrac{1}{2}r^2+\beta r+\frac{3}{2}\frac{\gamma}{E_{00}}+\frac{\gamma}{2r}\right) - \tfrac{1}{2}E_{00}\beta\,[-E_i(-E_{00}/r)+C+\ln E_{00}]\right\}, \tag{59}$$

$$J_2(r) = -\frac{4}{E_{00}^3}\left[\tfrac{1}{2}r^2+(\beta-\tfrac{1}{2}E_{00})\,r-\tfrac{1}{2}E_{00}\beta\ln r-\frac{\gamma}{r}+\frac{E_{00}\gamma}{4r^2}\right], \tag{60}$$

$$\alpha = \frac{E_{00}}{\exp(-E_{00})-1}, \quad \beta = \tfrac{1}{2}(E_{11}-3), \quad \gamma = \tfrac{1}{2}(E_{11}+1), \tag{61}$$

$$C = -\int_0^\infty (\ln t)\,e^{-t}\,dt = 0{\cdot}577215665\ldots \quad \text{(Euler–Mascheroni constant)},$$

$$E_i(-x) = -\int_x^\infty \frac{\exp(-x_1)}{x_1}\,dx_1 \quad \text{(exponential integral)}.$$

As $r \to \infty$

$$h_{11} \sim \frac{\alpha r}{E_{00}}\left\{\left[1-\frac{4\beta}{E_{00}}+\frac{6\gamma}{E_{00}^3}+\frac{E_{00}}{\alpha}(K_1+K_2)\right] - \frac{E_{00}}{r}\left[1-\frac{2\beta}{E_{00}}+\frac{3\gamma}{E_{00}^3}+\frac{E_{00}}{2\alpha}(K_1+K_2)\right]\right\} + O(1/r). \tag{62}$$

Applying (23) to (58) and matching (51) and (62) gives

$$G_4K_1 = G_1 - G_3J_1(1) + G_2[G_5 - J_2(1)], \tag{63}$$

$$-G_4K_2 = G_1 - G_2J_2(1) + G_3[G_5 - J_1(1)], \tag{64}$$

where $$G_1 = \frac{\overline{T}_{11}}{\overline{T}_{00} - \overline{T}_\infty - (\overline{Y}_O)_\infty}, \quad G_2 = \alpha \exp(-E_{00})(1 + \tfrac{1}{2}E_{00}),$$

$$G_3 = \alpha(1 - \tfrac{1}{2}E_{00}), \quad G_4 = (G_3 - G_2)/\alpha, \quad G_5 = \left(1 - \frac{4\beta}{E_{00}} + \frac{6\gamma}{E_{00}^3}\right)\Big/ E_{00}.$$

Equation (29) applied to (58) gives a relation for E_{11}:

$$[K_1 + \alpha J_1(1)] + [K_2 + \alpha J_2(1)] \exp(-E_{00})[1 + E_{00}(1 + \tfrac{1}{2}E_{00})] + G_3J_1'(1) + G_2J_2'(1) = \frac{\overline{L}mE_{11}}{\overline{T}_{00} - \overline{T}_\infty - (\overline{Y}_O)_\infty}. \tag{65}$$

The prime in (65) signifies differentiation with respect to r.

A uniformly valid expression for $\overline{p}$ is

$$\overline{p} = \overline{p}_0 + \epsilon(\overline{p}_1 + \overline{P}_0) + \frac{E_{00}}{\exp(-E_{00}) - 1}\left[\frac{1}{r} + \tfrac{1}{2}\epsilon(\mu - 1)\right] + o(\epsilon). \tag{66}$$

7. Modification of the thin-flame position

To find $r_*(\theta, \epsilon)$, one must solve $L\overline{q} = 0$ where $\overline{q} = \overline{Y}_F - \overline{Y}_O + (\overline{Y}_O)_\infty$. Again the method of inner and outer expansion is invoked, and in view of results of §§5 and 6

$$\overline{q^o}(\rho, \mu, \epsilon) = \sum_{n=0}^{\infty} g_n(\epsilon)\,\overline{Q}_n(\rho, \mu) = -\epsilon B_1 \frac{\exp[(\frac{1}{2}\rho)(\mu - 1)]}{\rho} - o(\epsilon), \tag{67}$$

$$\overline{q^i}(r, \mu, \epsilon) = \sum_{n=0}^{\infty} f_n(\epsilon)\,\overline{q}_n(r, \mu) = m_{00}(r) + \epsilon[m_{01}(r)P_0(\mu) + m_{11}(r)P_1(\mu)] + o(\epsilon). \tag{68}$$

It is easy to show

$$m_{00} = B_1[\exp(-E_{00}/r) - 1]/E_{00}, \quad B_1 = -E_{00}[\alpha_F + (\overline{Y}_O)_\infty]. \tag{69}$$

From equations (10), (11), (49), and (69) well-known results for the convection-free case (Williams 1965) are recovered:

$$\overline{Y}_F(1) = \alpha_F - \frac{\overline{L}m[\alpha_F + (\overline{Y}_O)_\infty]}{\overline{L}m + \overline{T}_\infty + (\overline{Y}_O)_\infty - \overline{T}_{00}} + O(\epsilon), \tag{70}$$

$$r_* = E_{00}/\ln[1 + \{(\overline{Y}_O)_\infty/\alpha_F\}] = \ln\left[1 + \frac{(\overline{Y}_O)_\infty + \overline{T}_\infty - \overline{T}_{00}}{\overline{L}m}\right]\Big/\ln[1 + \{(\overline{Y}_O)_\infty/\alpha_F\}]. \tag{71}$$

Analogous to (53), if M and N are constants of integration,

$$m_{01} = M + N\left(\frac{\exp(-E_{00}/r) - 1}{E_{00}}\right) - E_{01}[\alpha_F + (\overline{Y}_O)_\infty] \times \left[\exp(-E_{00}/r)\left(1 - \frac{1}{r}\right) + \frac{\exp(-E_{00}) - \exp(-E_{00}/r)}{E_{00}}\right]. \tag{72}$$

Matching to (67)

$$M = [\alpha_F + (\overline{Y}_O)_\infty]\left[E_{01}\left(1+\frac{\exp(-E_{00})-1}{E_{00}}\right)-\tfrac{1}{2}E_{00}\right]. \quad (73)$$

Applying (7) to (72) gives with the aid of (69)

$$N = E_{00}M - E_{01}[\alpha_F + (\overline{Y}_O)_\infty]\exp(-E_{00}). \quad (74)$$

Analogous to (58), if W and S are constants of integration,

$$m_{11} = [W + B_1 J_1(r)](r - \tfrac{1}{2}E_{00}) + [S + B_1 J_2(r)]\exp(-E_{00}/r)(r + \tfrac{1}{2}E_{00}). \quad (75)$$

Matching (75) to (67) and again applying (7) gives

$$G_6 W = A_{11} - (\exp(-E_{00}))A_{22}, \quad G_6 S = (1 - (E_{00}/\alpha)G_3)A_{22} - A_{11}, \quad (76)$$

where $$G_6 = 1 - (E_{00}/\alpha)G_3 - \exp(-E_{00}), \quad A_{22} = -B_1 G_5 \quad (77)$$

and $$A_{11} = -E_{11}[\alpha_F + (\overline{Y}_O)_\infty]\exp(-E_{00}) - (B_1/\alpha)[G_3 J_1'(1) + J_2'(1)G_2] - B_1 J_2(1)\exp(-E_{00}) - B_1 J_1(1)[1 - (E_{00}/\alpha)G_3]. \quad (78)$$

The uniformly valid solution for $\overline{q}$ is

$$\overline{q} = \overline{q}_0 + \epsilon(\overline{q}_1 + \overline{Q}_0) + B_1[(1/r) + \tfrac{1}{2}\epsilon(\mu - 1)] + o(\epsilon). \quad (79)$$

8. The Clausius–Clapeyron equation

Because equation (20) was used in place of (9), it is necessary to be certain that the adopted

$$\overline{T}(1,\mu,\epsilon) = \overline{T}_{00} + \epsilon(\overline{T}_{01} + \mu\overline{T}_{11}) + o(\epsilon) \quad (80)$$

is compatible with

$$\overline{Y}_F(1,\mu,\epsilon) = -(\overline{Y}_O)_\infty + m_{00}(1) + \epsilon[m_{01}(1) + \mu m_{11}(1)] + o(\epsilon). \quad (81)$$

Substituting these expansions into (9), expanding in ϵ, and then adopting a hopefully convergent iterative procedure, one arrives at the following for assigning improved values for $\overline{T}_{00}$, $\overline{T}_{01}$, and $\overline{T}_{11}$:

$$\overline{T}_{00}^{(n+1)} = \frac{\chi\overline{T}_B}{\chi - \overline{T}_B \ln[\{m_{00}^{(n)}(1) - (\overline{Y}_O)_\infty\}/\alpha_F]}, \quad (82)$$

$$\overline{T}_{01}^{(n+1)} = \frac{\overline{T}_{00}^{(n)2}}{\chi}\left[\frac{m_{01}^{(n)}(1)}{\{m_{00}^{(n)}(1) - (\overline{Y}_O)_\infty\}}\right], \quad (83)$$

$$\overline{T}_{11}^{(n+1)} = \frac{\overline{T}_{00}^{(n)2}}{\chi}\left[\frac{m_{11}^{(n)}(1)}{m_{00}^{(n)}(1) - (\overline{Y}_O)_\infty}\right]. \quad (84)$$

9. Extension to higher Reynolds numbers

It will now be shown that the results found for $R = 0$ may be extended unchanged at least to such finite R as obey $R = O(\epsilon)$ where $\epsilon \ll 1$.

For small R, Oseen (Lamb 1945) proposed a linearized version (2) of the Navier–Stokes equation as an adequate approximation. Proudman & Pearson (1957) later showed the Oseen equation did yield a uniformly valid lowest-order approximation to the Navier–Stokes equation, so satisfaction of boundary conditions at the spherical surface up to $O(R)$ had meaning, but more exact solution—though possible with enough ingenuity—was rigorously meaningless.

Accordingly, in view of

$$\epsilon u_r(1,\mu,\epsilon) = E_{00}+\epsilon(E_{01}+E_{11}P_1)+o(\epsilon), \tag{85}$$

solution to (12) is taken in the form

$$\psi = C_2(\mu-1)+\left(\frac{r^2}{2}+\frac{A_2}{r}\right)(1-\mu)+\frac{B_2}{R}(1+\mu)\{1-\exp[-\tfrac{1}{2}Rr(1-\mu)]\}, \tag{86}$$

where A_2, B_2, and C_2 are independent of R but not necessarily of ϵ. For $R \ll 1$, at $r\to 1$

$$\psi = C_2(\mu-1)+(\tfrac{1}{2}r^2+(A_2/r)+\tfrac{1}{2}B_2r)(1-\mu^2)+O(R). \tag{87}$$

If $C_2 = -E_0$, $A_2 = \frac{1}{4}(E_1+1)$, and $B_2 = \frac{1}{2}(E_1-3)$, then the Stokes solution (16) and the Oseen solution (86) are the same so far as the role they would play in the Schvab–Zeldovitch integrals $\bar{p}$ and $\bar{q}$ for terms $O(\epsilon)$ or larger.

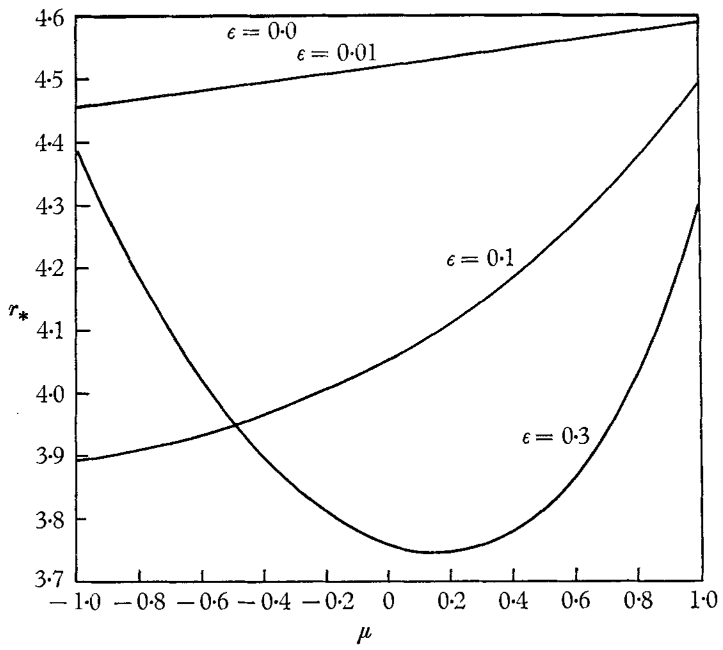

FIGURE 2. Modification of the thin-flame position as a function of angle for several values of the Peclet number ϵ.

10. Numerical calculations

As a typical case involving burning of a hydrocarbon droplet in a hot oxidizing gas, the following values were assigned: $m = 186$, $\alpha_F = 3{\cdot}207$, $\bar{T}_B = 0{\cdot}0593$, $\bar{T}_\infty = 0{\cdot}25$, $m_F = 58$, $(\bar{Y}_O)_\infty = 1{\cdot}685$, $\bar{L} = 0{\cdot}00169$, and $\bar{\chi} = 0{\cdot}491$. It is found after five iterations using (82)–(84) that $\bar{T}_{00} = 0{\cdot}0576$, $\bar{T}_{01} = -0{\cdot}1606\times 10^{-9}$, and $\bar{T}_{11} \simeq 0{\cdot}1424\times 10^{-9}$. Hence the surface temperature of the purely diffusive transport case, which is about 3 per cent below boiling, is absolutely negligibly modified by convection. In the Appendix it is shown that the ratio of the net mass-transfer rate with small convection to that without goes as $[1+\frac{1}{2}\epsilon+o(\epsilon)]$. Furthermore, it is found that $E_{00} = 1{\cdot}9422$, $E_{01} = (\frac{1}{2}E_{00})$, and $E_{11} = -0{\cdot}7145$.

The local mass-transfer rate increases linearly in μ from a minimum at the downstream point of the droplet to a maximum at the upstream point; for $\epsilon = 0{\cdot}1$, the variation is from $\epsilon u_r = 1{\cdot}9679$ to $\epsilon u_r = 2{\cdot}1108$. Because $E_{11} < 0$, the Appendix shows that the ratio of the drag for the flow with surface mass transfer to the drag without is 1·2382.

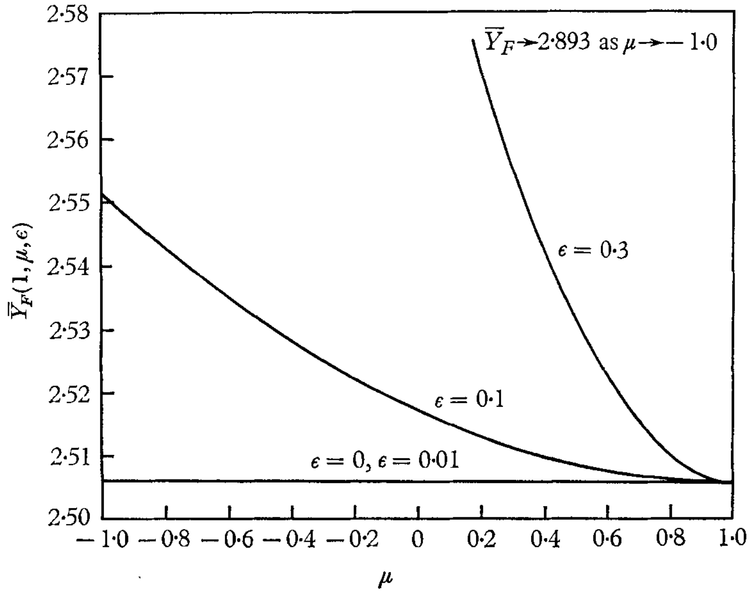

FIGURE 3. Variation in the stoichiometrically adjusted mass fraction of fuel $\bar{Y}_F$ with angle at the droplet surface.

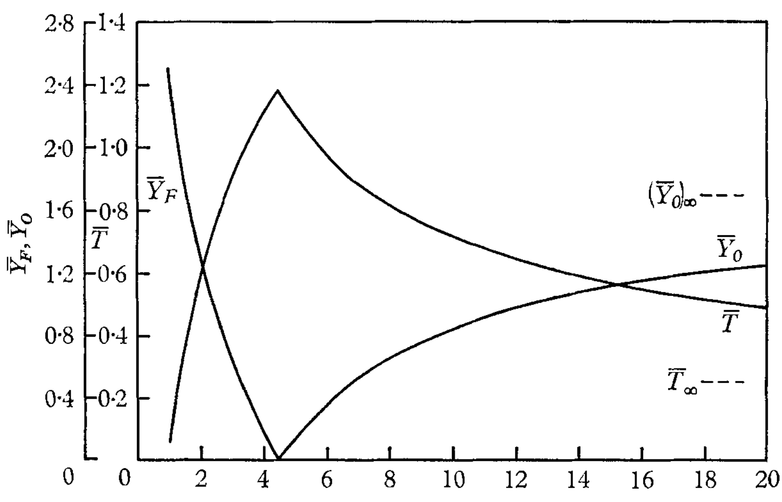

FIGURE 4. Profiles of the dependent variables with radial position for $\epsilon = 0{\cdot}1$ and $\mu = 1{\cdot}0$.

A plot of the thin-flame position for several values of ϵ (figure 2) reveals that the flame is drawn closer to the droplet, and given a wake-like distortion, as ϵ increases. An indication of the limit of validity of the theory may be revealed by the fact that as $\epsilon \uparrow 0{\cdot}3$ the point of closest proximity of the flame to the droplet moves downstream from the forward stagnation point. Further results indicating

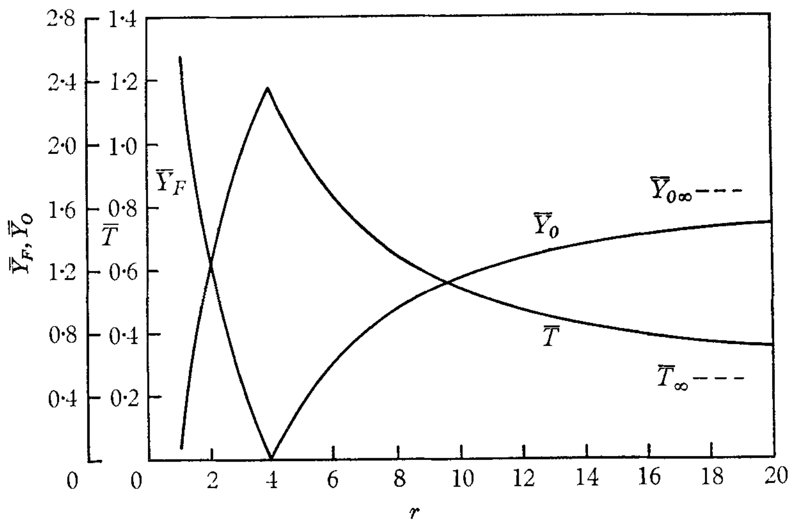

FIGURE 5. Profiles of the dependent variables with radial position for $\epsilon = 0{\cdot}1$ and $\mu = -1{\cdot}0$.

modifications because of convection in dependent-variable profiles are given in figures 3–5; the so-called adiabatic flame temperature $\overline{T}(r_*, \mu, \epsilon) = 1{\cdot}1802$ for all μ for all $\epsilon \ll 1$.

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Appendix. Net mass transfer and drag coefficient

If $\dot{m}$ is the net transfer of mass (dimensional) at the droplet surface

$$\dot{m} = 2\pi\rho a^2 U_\infty \int_{-1}^{1} u_r(1, \mu, \epsilon)\, d\mu. \tag{A 1}$$

Since
$$\int_{-1}^{1} P_m(\mu)\, d\mu = \begin{Bmatrix} 2 & m = 0, \\ 0 & m \neq 0, \end{Bmatrix} \tag{A 2}$$

$$\dot{m} = 4\pi\rho a D \sum_{n=0}^{\infty} E_{0n} f_n(\epsilon) = 4\pi\rho a D E_{00}[1 + (E_{01}/E_{00})\,\epsilon + o(\epsilon)]. \tag{A 3}$$

If $\overline{T}_{01} = 0$, from equation (57), $E_{01}/E_{00} = \frac{1}{2}$.

There is no net force on the droplet for the radially symmetric case $\epsilon = 0$. When $\epsilon \neq 0$, there is no lift because of the azimuthal symmetry and the drag coefficient is given by ($r = 1$)

$$C_D = \frac{F_x}{\rho U_\infty^2 a^2} = \frac{2\pi}{R}\int_{-1}^{1} [-PR\mu + \tau_{rr}\mu - \tau_{r\theta}(1-\mu^2)^{\frac{1}{2}}]\,du, \quad \text{(A 4)}$$

where F_x is the component of force parallel to the free-stream. Since

$$-\psi = \left(\frac{E_{00}}{\epsilon} + E_{01}\right)(\mu+1) + \left(r^2 + \frac{E_{11}+1}{2r} + \frac{E_{11}-3}{2}r\right)\left(\frac{\mu^2-1}{2}\right) + o(1), \quad \text{(A 5)}$$

and since P is a harmonic function bounded at infinity

$$P = P_0 + \frac{A}{r} + \frac{B}{r^2}\mu + o(1), \quad \text{(A 6)}$$

where P_0 is an arbitrary pressure datum. From equation (2), when $\partial \mathbf{u}/\partial x = 0$

$$A = 0, \quad B = (E_{11}-3)/2R. \quad \text{(A 7)}$$

Whence

$$C_D = (2\pi/R)(3 - E_{11}) + o(1/R). \quad \text{(A 8)}$$

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